Nonminimal Generalized Kane's Impulse–Momentum Relations

Abdulrahman H. Bajodah,* Dewey H. Hodges,† and Ye-Hwa Chen‡
Georgia Institute of Technology, Atlanta, Georgia 30332

Introduction

A WELL-KNOWN methodology for deriving equations of motion for dynamical systems is Kane's method. A close relationship between the method and those of D'Alembert and of Gibbs and Appell has been noted in the literature.[1,2] The impulse–momentum approach to modeling impact was adopted by Kane[3] in a general form for both holonomic and nonholonomic impulsive constraints. The basic assumption in the approach is that the duration of the impact is very short compared to the time interval of the motion, so that the impact can be considered a discrete event, and the change in the configuration of the system during impact is ignorable, although the changes in velocities of the system components can be significant.[4] This allows converting the differential equations that govern the dynamics of the system to algebraic equations, by integrating the equations in velocities over the infinitesimal time period of impact.

The relationships between the velocities before and after impact are given by an experimentally evaluated constant that is dependent on the material and the geometry of the collided bodies and the surfaces of collision, called the coefficient of restitution.[5]

The impulse–momentum approach to modeling impact was applied to different modeling methodologies. An example is the method of coordinate partitioning,[6] where the acceleration form of constraint equations is utilized to eliminate dependent acceleration variables in favor of independent acceleration variables to derive minimal equations of motion. Another example is the Hamilton equations of motion.[7]

The impulse–momentum approach was followed successfully in the area of multibody system dynamics to model the intermittent motion of both rigid[6] and flexible[8] systems, but it is interesting to notice that using the approach for multibody system dynamics in the context of Kane's equations of motion was not done until recently.[9]

In this Note, the impulse–momentum approach is extended to the nonminimal nonholonomic form by explicitly including the effect of nonholonomic constraints on the rapid changes of the generalized speeds.

The nonminimal form of the equations of motion provides a convenient way to analyze the intermittent motion of both nonholonomic systems and complex holonomic systems with numerous configuration settings but relatively low numbers of degrees of freedom. The latter case pertains to analyses in which pseudogeneralized coordinates (i.e., additional configuration variables) are needed to facilitate the formulation, and hence more holonomic constraints are added.

In the next section, nonholonomic generalized impulses and momenta are defined and are related to their holonomic counterparts by means of the constraint matrix.

Received 18 December 2003; revision received 23 June 2004; accepted for publication 1 July 2004.

*Graduate Student, School of Aerospace Engineering; gt5158a@prism.gatech.edu.

†Professor, School of Aerospace Engineering; dewey.hodges@ae.gatech.edu. Fellow AIAA.

‡Professor, George W. Woodruff School of Mechanical Engineering; yehwa.chen@me.gatech.edu.

Generalized Impulse and Momentum

An inertial reference frame $\mathcal{R}$ is considered, in which n generalized coordinates are used to describe the configuration of a set of ν particles and μ rigid bodies forming a nonholonomic system S possessing p degrees of freedom. Let $\boldsymbol{R}_i$ be the resultant active force on the ith particle, P_i. The resultant active forces on the ith rigid body, B_i, are equivalent to a force $\boldsymbol{Z}_i$ on a point Q_i on B_i, together with a torque $\boldsymbol{T}_i$. Also, let t_1 and t_2 be the initial and final instants of time that are close enough so that the generalized coordinates $q_1 \ldots q_n$ can be considered as constants throughout the interval bounded by t_1 and t_2. The rth nonholonomic generalized impulse $\tilde{\mathcal{I}}_r$ is defined as the integral of the rth generalized active force over $[t_1, t_2]$ (see Ref. 4):

$$\tilde{\mathcal{I}}_r(q, u, t_1) = \int_{t_1}^{t_2} \tilde{F}_r(q, u, t)\,\mathrm{d}t, \qquad r = 1, \ldots, p \tag{1}$$

$$= \sum_{i=1}^{\nu} \int_{t_1}^{t_2} \tilde{\boldsymbol{v}}_r^{P_i} \cdot \boldsymbol{R}_i\,\mathrm{d}t + \sum_{i=1}^{\mu} \int_{t_1}^{t_2} \tilde{\boldsymbol{v}}_r^{Q_i} \cdot \boldsymbol{Z}_i\,\mathrm{d}t + \sum_{i=1}^{\mu} \int_{t_1}^{t_2} \tilde{\boldsymbol{\omega}}_r^{B_i} \cdot \boldsymbol{T}_i\,\mathrm{d}t \tag{2}$$

where $\tilde{\boldsymbol{v}}_r^{P_i}$, $\tilde{\boldsymbol{v}}_r^{Q_i}$ are the rth nonholonomic partial velocities of P_i and Q_i, respectively, and $\tilde{\boldsymbol{\omega}}_r^{B_i}$ is the rth nonholonomic partial angular velocity of B_i (see Ref. 4). Alternatively, the rth component of the holonomic generalized impulse can be defined for the system S using the holonomic partial velocities of P_i and Q_i and the holonomic partial angular velocity of B_i, denoted by $\boldsymbol{v}_r^{P_i}$, $\boldsymbol{v}_r^{Q_i}$, and $\boldsymbol{\omega}_r^{B_i}$, respectively. It takes the same expression as Eq. (2) with the tildes removed from above the symbols, where $r = 1, \ldots, n$ (see Ref. 4). The system is assumed to be simple nonholonomic; that is, the generalized speeds satisfy the relations

$$u_{p+r} = \sum_{s=1}^{p} A_{rs} u_s + B_r, \qquad r = 1, \ldots, n - p \tag{3}$$

where the scalars A_{rs} and B_r are functions of the generalized coordinates $q_1, \ldots, q_n$, and t. By using the relation between the holonomic and nonholonomic partial velocities and partial angular velocities given by[4]

$$^{\mathcal{R}}\tilde{\boldsymbol{v}}_r^{P} = {}^{\mathcal{R}}\boldsymbol{v}_r^{P} + \sum_{s=1}^{n-p} {}^{\mathcal{R}}\boldsymbol{v}_{p+s}^{P} A_{sr}(q, t), \qquad r = 1, \ldots, p \tag{4}$$

$$^{\mathcal{R}}\tilde{\boldsymbol{\omega}}_r^{B} = {}^{\mathcal{R}}\boldsymbol{\omega}_r^{B} + \sum_{s=1}^{n-p} {}^{\mathcal{R}}\boldsymbol{\omega}_{p+s}^{B} A_{sr}(q, t), \qquad r = 1, \ldots, p \tag{5}$$

Equation (1) can be written as

$$\tilde{\mathcal{I}}_r(q, u, t_1) = \int_{t_1}^{t_2} \left[F_r(q, u, t) + \sum_{s=1}^{n-p} F_{p+s}(q, u, t) A_{sr}(q, t) \right] \mathrm{d}t \tag{6}$$

$$= \mathcal{I}_r(q, u, t_1) + \sum_{s=1}^{n-p} \mathcal{I}_{p+s}(q, u, t_1) A_{sr}(q, t), \qquad r = 1, \ldots, p \tag{7}$$

Remark: If the dynamical system is holonomic or constrained by simple nonholonomic constraints, then the constraint matrix A is only dependent on the generalized coordinates $q_1, \ldots, q_n$ and t and is independent of the generalized speeds $u_1, \ldots, u_n$. This implies that A can be regarded as constant in the interval $[t_1, t_2]$ during the evaluation of the generalized impulses and can be taken to be its value at the interval entry, $t = t_1$.

The rth nonholonomic generalized momentum $\tilde{\mathcal{P}}_r$ is defined as

$$\tilde{\mathcal{P}}_r(q,u) = \sum_{i=1}^{\nu} \tilde{\boldsymbol{v}}_r^{P_i} \cdot \boldsymbol{L}^{P_i} + \sum_{i=1}^{\mu} \tilde{\boldsymbol{v}}_r^{B_i^\star} \cdot \boldsymbol{L}^{B_i} + \sum_{i=1}^{\mu} \tilde{\boldsymbol{\omega}}_r^{B_i} \cdot \boldsymbol{H}^{B_i} \qquad r = 1, \ldots, p \tag{8}$$

where $\boldsymbol{L}^{P_i}, \boldsymbol{L}^{B_i}$, and $\boldsymbol{H}^{B_i}$ are respectively the linear momentum of the ith particle, the linear momentum of the ith body, and the angular momentum of the ith body of the system. Therefore,

$$\tilde{\mathcal{P}}_r(q,u) = \sum_{i=1}^{\nu} m_{P_i} \tilde{\boldsymbol{v}}_r^{P_i} \cdot \boldsymbol{v}^{P_i}(t) + \sum_{i=1}^{\mu} m_{B_i} \tilde{\boldsymbol{v}}_r^{B_i^\star} \cdot \boldsymbol{v}^{B_i^\star}(t) + \sum_{i=1}^{\mu} \tilde{\boldsymbol{\omega}}_r^{B_i} \cdot \underline{\mathbf{I}}^{B_i} \cdot \boldsymbol{\omega}^{B_i}(t), \qquad r = 1, \ldots, p \tag{9}$$

where m_{P_i} is the mass of the ith particle P_i, m_{B_i} is the mass of the ith body B_i, $B_i^\star$ is its center of mass, and $\underline{\mathbf{I}}^{B_i}$ is its central inertia dyadic.

The use of the full set of generalized speeds in the expressions of velocities and angular velocities of the particles and bodies composing a nonholonomic system makes it feasible to introduce the rth holonomic generalized momentum $\mathcal{P}_r$, which can take the same expression (9) with the tildes removed from above the partial velocities and partial angular velocities, and with $r = 1, \ldots, n$. Similar to the generalized impulse, the relations (4) and (5) between the holonomic and nonholonomic partial angular velocities and partial velocities can be used to obtain the nonminimal representation of the nonholonomic generalized momentum

$$\tilde{\mathcal{P}}_r(q,u) = \mathcal{P}_r(q,u) + \sum_{s=1}^{n-p} \mathcal{P}_{p+s}(q,u) A_{sr}(q,t), \qquad r = 1, \ldots, p \tag{10}$$

Nonminimal Impulse–Momentum Relations

Both the holonomic and the nonholonomic generalized momenta can be related to the generalized inertia forces of system. The rth generalized inertia force of the particles and bodies composing a nonholonomic system is given by

$$\tilde{F}_r^\star(q,u,\dot{u},t) = -\sum_{i=1}^{\nu} \tilde{\boldsymbol{v}}_r^{P_i} \cdot \frac{\mathrm{d}\boldsymbol{L}^{P_i}}{\mathrm{d}t} - \sum_{i=1}^{\mu} \tilde{\boldsymbol{v}}_r^{B_i^\star} \cdot \frac{\mathrm{d}\boldsymbol{L}^{B_i}}{\mathrm{d}t} - \sum_{i=1}^{\mu} \tilde{\boldsymbol{\omega}}_r^{B_i} \cdot \frac{\mathrm{d}\boldsymbol{H}^{B_i}}{\mathrm{d}t}, \qquad r = 1, \ldots, p \tag{11}$$

If this expression of $\tilde{F}_r^\star$ is integrated from t_1 to t_2, with $t_1 \approx t_2$ such that the configuration of the system is invariant in the period $[t_1, t_2]$, then the partial angular velocities and partial velocities of the particles and bodies comprising the system are considered to be constants, as these quantities are dependent on the generalized coordinates, and independent of the generalized speeds. Therefore, Eq. (8) can be used to write this integral as

$$\int_{t_1}^{t_2} \tilde{F}_r^\star(q,u,\dot{u},t)\,\mathrm{d}t = \tilde{\mathcal{P}}_r[q,u(t_1)] - \tilde{\mathcal{P}}_r[q,u(t_2)] \qquad r = 1, \ldots, p \tag{12}$$

The equations of motion for impulsive motion are

$$\int_{t_1}^{t_2} \left[\tilde{F}_r(q,u,t) + \tilde{F}_r^\star(q,u,\dot{u},t)\right] \mathrm{d}t = 0, \qquad r = 1, \ldots, p \tag{13}$$

From Eqs. (1) and (2), these equations become

$$\tilde{\mathcal{I}}_r(q,u,t_1) = \tilde{\mathcal{P}}_r[q,u(t_2)] - \tilde{\mathcal{P}}_r[q,u(t_1)], \qquad r = 1, \ldots, p \tag{14}$$

From Eqs. (7) and (10), Eqs. (14) can be written as

$$\mathcal{P}_r[q,u(t_2)] + \sum_{s=1}^{n-p} \mathcal{P}_{p+s}[q,u(t_2)] A_{sr}(q,t_1) = \mathcal{P}_r[q,u(t_1)] + \mathcal{I}_r(q,u,t_1) + \sum_{s=1}^{n-p} \{\mathcal{P}_{p+s}[q,u(t_1)] + \mathcal{I}_{p+s}(q,u,t_1)\} A_{sr}(q,t_1), \qquad r = 1, \ldots, p \tag{15}$$

Constraints Effect on Impulsive Motion

Let

$$u = \lfloor u_1 \cdots u_n \rfloor^T = \left\lfloor u_I^T \quad u_D^T \right\rfloor^T \tag{16}$$

where $u_I = \lfloor u_1 \cdots u_p \rfloor^T$ and $u_D = \lfloor u_{p+1} \cdots u_n \rfloor^T$. Then, the matrix representation of Eq. (3) is

$$u_D = A(q,t) u_I + B(q,t) \tag{17}$$

where $A \in \mathbb{R}^{(n-p)\times p}$, $B \in \mathbb{R}^{n-p}$. Equation (17) can be written as

$$A_1(q,t) u = B(q,t) \tag{18}$$

where

$$A_1 = [-A \quad I] \tag{19}$$

To obtain the effect of constraints on the impulsive motion, Eq. (18) is evaluated at t_1 and t_2, and the evaluations are subtracted from each other, which yields

$$A_1[q(t_2),t_2] u(t_2) - A_1[q(t_1),t_1] u(t_1) = B[q(t_2),t_2] - B[q(t_1),t_1] \tag{20}$$

It is assumed here that the matrices A and B depend on time only implicitly through $q(t)$. This implies that the effect of constraints during the impulsive motion of the dynamical system is such that

$$A_1[q(t_2)] u(t_2) - A_1[q(t_1)] u(t_1) = B[q(t_2)] - B[q(t_1)] \tag{21}$$

Because the configuration is assumed to be unchanged in the time period $[t_1, t_2]$, the right-hand side of this equation vanishes, resulting in

$$A_1[q(t)] u(t) = c \tag{22}$$

where c is a constant that can be determined from the values of the generalized coordinates and the generalized speeds at the beginning of the time interval of the impulsive action.

Impulsive Dynamical Equations of Motion

The nonminimal form of the impulsive equations of motion can be obtained by using Eq. (15) with Eq. (22). It is noticed that Eq. (15) can be put in the form

$$A_2(q) \mathcal{P}_r[q,u(t_2)] = A_2(q)\{\mathcal{P}_r[q,u(t_1)] + \mathcal{I}_r(q,u,t_1)\} \qquad r = 1, \ldots, n \tag{23}$$

where

$$A_2 = [I \quad A^T] \tag{24}$$

$F^\star$ can be written in the form[10]

$$F^\star = -Q(q,t)\dot{u} - L(q,u,t) \tag{25}$$

where Q is a symmetric positive definite matrix. Integrating the expression (25) of $F^\star$ from t_1 to t_2, and using Eq. (12), the following equality is obtained:

$$\mathcal{P}[q,u(t_1)] - \mathcal{P}[q,u(t_2)] = \int_{t_1}^{t_2} F^\star\,\mathrm{d}t = -\int_{t_1}^{t_2} [Q(q,t)\dot{u} + L(q,u,t)]\,\mathrm{d}t \tag{26}$$

The second term on the right-hand side is negligible because it is an integral of a continuous function over the infinitesimal time interval $[t_1, t_2]$. Hence, Eq. (26) becomes

$$\mathcal{P}[q, u(t_1)] - \mathcal{P}[q, u(t_2)] = Q(q, t)[u(t_1) - u(t_2)] \tag{27}$$

Substituting Eq. (27) into Eq. (23) yields

$$A_2(q)Q(q, t)[u(t_2) - u(t_1)] = A_2(q)\mathcal{I}(q, u, t_1) \tag{28}$$

Equations (22) and (28) form the matrix system

$$\begin{bmatrix} A_1(q) \\ A_2(q)Q(q, t_1) \end{bmatrix}[u(t_2) - u(t_1)] = \begin{bmatrix} \mathbf{0} \\ A_2(q) \end{bmatrix}\mathcal{I}(q, u, t_1) \tag{29}$$

Remark: The coefficient matrix in Eq. (29) is the same as the matrix T in Ref. 10. The invertibility of T is guaranteed for all configurations and velocities that render the elements of the constraint matrix A finite. Therefore,

$$u(t_2) = T^{-1}(q, t_1)E(q, u, t_1) \tag{30}$$

where

$$T = \begin{bmatrix} A_1(q) \\ A_2(q)Q(q, t_1) \end{bmatrix} \tag{31}$$

$$E = \begin{bmatrix} A_1(q)u(t_1) \\ A_2(q)[Q(q, t_1)u(t_1) + \mathcal{I}(q, u, t_1)] \end{bmatrix} \tag{32}$$

Example: Four-Bar Linkage

The four-bar linkage shown in Fig. 1 moves in the vertical plane and has the dimensions $L_1 = 1.0$ m, $L_2 = 3.0$ m, $L_3 = L_4 = 2.5$ m. The linkage OQ is fixed to the inertial frame $\mathcal{R}$. Two particles P_1 and P_2 have masses m_1 and m_2, respectively, and are fixed to two vertices of the linkage, as shown in the figure. The mechanism is at rest at $q_1 = 0.2$ rad when the particle P_1 is struck by a particle P_3 that has mass m_3 and preimpact velocity $\boldsymbol{v}^{P_3}(t_1) = 5\boldsymbol{j}$ m/s. The coefficient of restitution for this collision is $e = 0.8$. It is required to find the resulting impulse forces and the changes in the velocities of the mechanism due to the impulsive action.

The mechanism has one degree of freedom. Nevertheless, three configuration variables, q_1, q_2, and q_3, are defined for convenience, as shown in the figure. The generalized speeds are defined as

$$u_1 = \dot{q}_1 L_1 \tag{33}$$

$$u_i = \dot{q}_i, \qquad i = 2, 3 \tag{34}$$

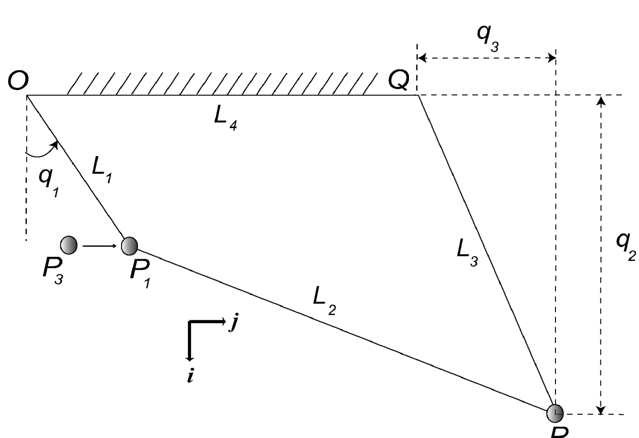

Fig. 1 Schematic for four-bar linkage.

The velocities of the two particles composing the mechanism relative to the inertial frame $\mathcal{R}$ are

$$^{\mathcal{R}}\boldsymbol{v}^{P_1} = u_1(\cos q_1 \boldsymbol{j} - \sin q_1 \boldsymbol{i}) \tag{35}$$

$$^{\mathcal{R}}\boldsymbol{v}^{P_2} = u_2\boldsymbol{i} + u_3\boldsymbol{j} \tag{36}$$

such that the partial velocities of the particles are

$$^{\mathcal{R}}\boldsymbol{v}_1^{P_1} = \cos q_1 \boldsymbol{j} - \sin q_1 \boldsymbol{i}, \qquad ^{\mathcal{R}}\boldsymbol{v}_2^{P_1} = {}^{\mathcal{R}}\boldsymbol{v}_3^{P_1} = 0 \tag{37}$$

$$^{\mathcal{R}}\boldsymbol{v}_1^{P_2} = 0, \qquad ^{\mathcal{R}}\boldsymbol{v}_2^{P_2} = \boldsymbol{i}, \qquad ^{\mathcal{R}}\boldsymbol{v}_3^{P_2} = \boldsymbol{j} \tag{38}$$

The corresponding accelerations relative to $\mathcal{R}$ are

$$^{\mathcal{R}}\boldsymbol{a}^{P_1} = \left[-\dot{u}_1 \sin q_1 - \left(u_1^2/L_1\right)\cos q_1\right]\boldsymbol{i} + \left[\dot{u}_1 \cos q_1 - \left(u_1^2/L_1\right)\sin q_1\right]\boldsymbol{j} \tag{39}$$

$$^{\mathcal{R}}\boldsymbol{a}^{P_2} = \dot{u}_2\boldsymbol{i} + \dot{u}_3\boldsymbol{j} \tag{40}$$

The generalized inertia forces are

$$-m_1{}^{\mathcal{R}}\boldsymbol{a}^{P_1} \cdot {}^{\mathcal{R}}\boldsymbol{v}_1^{P_1} - m_2{}^{\mathcal{R}}\boldsymbol{a}^{P_2} \cdot {}^{\mathcal{R}}\boldsymbol{v}_1^{P_2} = -m_1\dot{u}_1 \tag{41}$$

$$-m_1{}^{\mathcal{R}}\boldsymbol{a}^{P_1} \cdot {}^{\mathcal{R}}\boldsymbol{v}_2^{P_1} - m_2{}^{\mathcal{R}}\boldsymbol{a}^{P_2} \cdot {}^{\mathcal{R}}\boldsymbol{v}_2^{P_2} = -m_2\dot{u}_2 \tag{42}$$

$$-m_1{}^{\mathcal{R}}\boldsymbol{a}^{P_1} \cdot {}^{\mathcal{R}}\boldsymbol{v}_3^{P_1} - m_2{}^{\mathcal{R}}\boldsymbol{a}^{P_2} \cdot {}^{\mathcal{R}}\boldsymbol{v}_3^{P_2} = -m_2\dot{u}_3 \tag{43}$$

Therefore, the matrix Q in Eq. (29) is diagonal, and its diagonal elements are m_1, m_2, and m_2, respectively. Two holonomic constraint equations relating the generalized coordinates are

$$q_2^2 + q_3^2 - L_3^2 = 0 \tag{44}$$

$$L_4 + q_3 - \sqrt{L_2^2 - (q_2 - L_1 \cos q_1)^2} - L_1 \sin q_1 = 0 \tag{45}$$

After the constraints are differentiated and letting $u_I = u_1$ and $u_D = \lfloor u_2 \quad u_3 \rfloor^T$, the constraint matrices A and B of Eq. (17) are

$$A = \frac{\cos q_1 - X \sin q_1}{q_2 - Xq_3}\begin{Bmatrix} -q_3 \\ q_2 \end{Bmatrix} \tag{46}$$

$$B = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{47}$$

where

$$X = \frac{q_2 - L_1 \cos q_1}{\sqrt{L_2^2 - (q_2 - L_1 \cos q_1)^2}} \tag{48}$$

The given geometric condition is $q_1 = 0.2$ rad. The remaining generalized coordinates are found from Eqs. (44) and (45) to be

$$q_2 = -1.9331973 \text{ m} \tag{49}$$

$$q_3 = -1.5851649 \text{ m} \tag{50}$$

The corresponding value of X is -4.067863, and the constraint matrix A is

$$A = \begin{Bmatrix} -0.3382039 \\ 0.4124586 \end{Bmatrix} \tag{51}$$

The impulse force exerted on the particle P_2 is

$$\boldsymbol{R} = R\boldsymbol{j} \tag{52}$$

Therefore, the holonomic generalized impulses are

$$\mathcal{I}_1 = \boldsymbol{R} \cdot {}^{\mathcal{R}}\boldsymbol{v}_1^{P_1} = R \cos q_1 = 0.9801R \tag{53}$$

$$\mathcal{I}_2 = \boldsymbol{R} \cdot {}^{\mathcal{R}}\boldsymbol{v}_2^{P_1} = 0 \tag{54}$$

$$\mathcal{I}_3 = \boldsymbol{R} \cdot {}^{\mathcal{R}}\boldsymbol{v}_3^{P_1} = 0 \tag{55}$$

Specify $m_1 = m_2 = 1.0$ kg, and given that the entry conditions for the generalized speeds are $u_i(t_1) = 0$, $i = 1 \ldots 3$, the matrices T and E are

$$T = \begin{bmatrix} 0.3382039 & 1 & 0 \\ -0.4124586 & 0 & 1 \\ 1 & -0.3382039 & 0.4124586 \end{bmatrix} \tag{56}$$

$$E = \begin{Bmatrix} 0 \\ 0 \\ 0.9801R \end{Bmatrix} \tag{57}$$

To apply Eq. (30), it is necessary to solve for R. The coefficient of restitution e can be used to obtain the relative outlet velocities of the particles P_1 and P_3 involved in the collision in the $\boldsymbol{j}$ direction as

$$\left[{}^{\mathcal{R}}\boldsymbol{v}^{P_3}(t_2) - {}^{\mathcal{R}}\boldsymbol{v}^{P_1}(t_2)\right]\cdot\boldsymbol{j} = -e\left[{}^{\mathcal{R}}\boldsymbol{v}^{P_3}(t_1) - {}^{\mathcal{R}}\boldsymbol{v}^{P_1}(t_1)\right]\cdot\boldsymbol{j} \tag{58}$$

Letting ${}^{\mathcal{R}}\boldsymbol{v}^{P_3} = u_4\boldsymbol{j}$, this equation at $q_1 = 0.2$ rad becomes

$$u_4(t_2) - 0.9801u_1(t_2) = -0.8(5) = -4 \tag{59}$$

Setting $m_3 = 1.0$ kg, the impulse–momentum relation for the particle P_3 is

$$-R = m_3u_4(t_2) - m_3u_4(t_1) = u_4(t_2) - 5 \tag{60}$$

By eliminating $u_4(t_2)$ from Eqs. (59) and (60), R can be represented in terms of $u_1(t_2)$ as

$$R = 9 - 0.9801u_1(t_2) \tag{61}$$

Therefore, the column matrix E can be written as

$$E = \begin{Bmatrix} 0 \\ 0 \\ 0.9801R \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 8.8209 \end{Bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0.9801^2 & 0 & 0 \end{bmatrix} u(t_2) \tag{62}$$

The generalized speeds just after the action of the impulsive force are

$$u(t_2) = T^{-1}E \tag{63}$$

By substituting the expression (62) for E into Eq. (63) and solving for $u(t_2)$, the outlet generalized speeds are found to be $u_1(t_2) = 3.9290$ m/s, $u_2(t_2) = -1.3288$ m/s, $u_3(t_2) = 1.6205$ m/s. The impulse R is found from Eq. (61) to be 5.1492 N.

It is possible to use different numbers of configuration variables to derive different forms of the impulse–momentum equations. However, the complexity of the resulting equations increases as the number of configuration variables decreases. For example, it becomes very difficult to use one generalized coordinate to derive a minimal impulse–momentum equation for this one-degree-of-freedom mechanism. Nevertheless, to preserve minimality in the number of impulse–momentum equations, the mechanism may be treated as simple nonholonomic by considering the same generalized coordinates and eliminating two of the generalized speeds in favor of the third. Therefore, if the generalized speeds are chosen as Eqs. (33) and (34), then the velocities of P_1 and P_2 relative to $\mathcal{R}$ are given by Eqs. (35) and (36). Letting

$$u_I = u_1 \tag{64}$$

$$u_D = \lfloor u_2 \quad u_3 \rfloor^T \tag{65}$$

and using the expressions (46) and (47) for the matrices A and B to eliminate u_2 and u_3, ${}^{\mathcal{R}}\boldsymbol{v}^{P_2}$ becomes

$${}^{\mathcal{R}}\boldsymbol{v}^{P_2} = u_1\frac{\cos q_1 - X\sin q_1}{q_2 - Xq_3}(-q_3\boldsymbol{i} + q_2\boldsymbol{j}) \tag{66}$$

Hence, the nonholonomic partial velocities of the particles are

$${}^{\mathcal{R}}\tilde{\boldsymbol{v}}_1^{P_1} = \cos q_1\boldsymbol{j} - \sin q_1\boldsymbol{i} \tag{67}$$

$${}^{\mathcal{R}}\tilde{\boldsymbol{v}}_1^{P_2} = \frac{\cos q_1 - X\sin q_1}{q_2 - Xq_3}(-q_3\boldsymbol{i} + q_2\boldsymbol{j}) \tag{68}$$

The nonholonomic generalized momenta before and after the impact are

$$\tilde{\mathcal{P}}_1(t_1) = 0 \tag{69}$$

$$\begin{aligned}\tilde{\mathcal{P}}_1(t_2) &= m_1{}^{\mathcal{R}}\boldsymbol{v}^{P_1}\cdot{}^{\mathcal{R}}\tilde{\boldsymbol{v}}_1^{P_1} + m_2{}^{\mathcal{R}}\boldsymbol{v}^{P_2}\cdot{}^{\mathcal{R}}\tilde{\boldsymbol{v}}_1^{P_2} \\ &= m_1u_1(t_2) + m_2u_1(t_2)\left(\frac{\cos q_1 - X\sin q_1}{q_2 - Xq_3}\right)^2\left(q_2^2 + q_3^2\right)\end{aligned} \tag{70}$$

and the nonholonomic generalized impulse is

$$\tilde{I}_1 = \boldsymbol{R}\cdot{}^{\mathcal{R}}\tilde{\boldsymbol{v}}_1^{P_1} = R\cos q_1 \tag{71}$$

Evaluating Eqs.(69–71) at $q_1 = 0.2$ rad., the minimal generalized impulse–momentum equation

$$\tilde{\mathcal{P}}_1(t_2) - \tilde{\mathcal{P}}_1(t_1) = \tilde{I}_1$$

becomes

$$1.2845u_1(t_2) = 0.9801R \tag{72}$$

Solving this equation together with Eq. (61) for $u_1(t_2)$ and R and applying the constraint equation, Eq. (17), to solve for u_2 and u_3 yield the same values of the outlet generalized speeds and the impulse force obtained above.

The two mentioned methods for writing impulse–momentum equations are comparable in the required effort and in complexity of the resulting equations. However, writing all exit generalized speeds (at $t = t_2$) as an explicit vector field that is dependent on generalized coordinates, generalized impulses, and all inlet generalized speeds (at $t = t_1$) is more beneficial whenever the Jacobian of this vector field is needed. An example is minimizing a function of exit generalized speeds over the admissible configuration settings for some known values of inlet generalized speeds and generalized momenta.

Summary

Based on the assumption of unchanging generalized coordinates during the action of impulsive forces, the resulting nonminimal Kane's impulse–momentum equations appear in an algebraic form. The equations can be used either to study the effects of impacts or to study the effects of sudden activations of holonomic and/or nonholonomic constraints. In the first case, the coefficient of restitution is used to provide an additional kinematic relation to solve for the impulse. In the second case, the kinematic relation is provided by the equation of the constraint that is being activated. In all cases, the equations are used in a discrete manner to describe the "no-transient-time" change in velocities.

The nonminimal impulse–momentum equations form an alternative way to study impulsive constraints and gain the advantage of simplicity over the equations presented in Ref. 4 if the difference between the number of configuration and motion parameters is large. It also eliminates the need of using nonholonomic partial velocities to simplify the resulting equations. The explicit representation of all exit generalized speeds in terms of quantities that are dependent on all inlet generalized speeds is particularly beneficial in studying the effects of configuration settings, initial velocities, and impulsive loadings on the velocities of various parts of mechanisms and structures.

References

[1] Kane, T. R., and Wang, C. F., "On the Derivation of Equations of Motion," *Journal of the Society of Industrial and Applied Mathematics*, Vol. 13, No. 2, 1965, pp. 487–492.

[2]Sharf, I., D'Eleuterio, G. M. T., and Hughes, P. C., "On the Dynamics of Gibbs, Appell, and Kane," *European Journal of Mechanics. A, Solids*, Vol. 11, No. 2, 1992, pp. 145–155.
[3]Kane, T., "Impulsive Motions," *Journal of Applied Mechanics*, Vol. 29, No. 4, 1962, pp. 715–718.
[4]Kane, T. R., and Levinson, D. A., *Dynamics: Theory and Applications*, 1st ed., McGraw–Hill Series in Mechanical Engineering, McGraw–Hill, New York, 1985.
[5]Goldsmith, W., *Impact*, Edward Arnold, London, 1960.
[6]Haug, E., Wehage, R., and Barman, N., "Design Sensitivity Analysis of Planar Mechanism and Machine Dynamics," *Journal of Mechanical Design*, Vol. 103, No. 3, 1981, pp. 560–570.
[7]Lankarani, H., and Nikaravesh, P., "Canonical Impulse–Momentum Equations for Impact Analysis of Multibody Systems," *Journal of Mechanical Design*, Vol. 114, No. 1, 1992, pp. 180–186.
[8]Khulief, Y., and Shabana, A., "Dynamic Analysis of Constrained Systems of Rigid and Flexible Bodies with Intermittent Motion," *Journal of Mechanisms, Transmissions, and Automation in Design*, Vol. 108, No. 1, 1986, pp. 38–44.
[9]Chang, C.-C., Liu, C., and Huston, R., "Dynamics of Multibody Systems Subjected to Impulsive Constraints," *Multibody System Dynamics*, Vol. 8, No. 2, 2002, pp. 161–184.
[10]Bajodah, A. H., Hodges, D. H., and Chen, Y. H., "New Form of Kane's Equations of Motion for Constrained Systems," *Journal of Guidance, Control, and Dynamics*, Vol. 26, No. 1, 2003, pp. 79–88.

Differentiator-Free Nonlinear Proportional-Integral Controllers for Rigid-Body Attitude Stabilization

Kamesh Subbarao*
The University of Texas at Arlington, Arlington, Texas 76019
and
Maruthi R. Akella†
The University of Texas at Austin, Austin, Texas 78712

I. Introduction

THE past two decades have witnessed several important developments in the design of feedback controllers for rigid-body attitude stabilization (set-point regulation) and maneuver tracking objectives.[1,2] However, there still remain certain open problems in this field that are of great theoretical and practical interest. In particular, from the standpoint of external disturbance torque rejection, there currently exists no unified framework for designing simple control structures inspired from linear control theory such as proportional-integral (PI), proportional-integral-derivative (PID), and their variants. The main hindrance, of course, stems from the fact that the governing differential equations for the kinematics and dynamics of rigid-body attitude motion are nonlinear in nature.

In this Note, we present a novel framework of constructing PI-like controllers for attitude stabilization without using body angular rate (velocity) measurements. Our interest in the velocity-free control framework is largely motivated from practical considerations wherein the assumption of availability of the angular velocity measurement is not always satisfied because of either cost limitations or implementation constraints (e.g., tachometers are not available with all robotic manipulators). The earliest known result in the field of velocity-free attitude control was given by Lizarralde and Wen[3] through the passivity framework. This result utilizes the Euler parameter (quaternion) kinematics and a passivity filter whose state enables the construction of the attitude stabilizing controller. The structure of the filter vector state is shown to be governed through a stable first-order linear differential equation driven by the vector part of the quaternion (attitude measurement). Interestingly, the passivity filter does not have high-pass characteristics and in theory admits arbitrarily slow bandwidth. Also, for sufficiently small frequencies in the filter's input signal, the filter output approximates a pseudo-velocity-like state. However, this is not true for input signals with higher magnitudes of frequencies.

Subsequent extensions to this velocity-free controller framework were presented by Tsiotras[4] for kinematics expressed in terms of the nonredundant Gibbs vector and the modified Rodrigues parameters vector sets. Further generalizations to the case of attitude tracking along prescribed reference trajectories were given by Akella[5] and Akella and Kotamraju.[6]

One important merit possessed by all of these passivity-based schemes is that the underlying velocity-free control law is independent of the inertia matrix for the case of set-point regulation, thereby automatically providing stability robustness in the presence of arbitrarily large inertia parameter errors. At the same time, none of the aforementioned results provide room for integral control action that can potentially help eliminate or reduce steady-state attitude error in the presence of constant external disturbance torques. In a recent development, Subbarao[7] approached the attitude stabilization problem from the nonlinear PID control perspective wherein the full-state vector including the angular velocity was employed for feedback purposes. Even though the control scheme requires full-state feedback when compared to the passivity-filter based constructions, this result brings the useful provision for integral feedback. In terms of robustness with respect to inertia parameter uncertainties, the PID construction of Subbarao[7] is nearly independent of the inertia matrix in the sense that it implicitly requires only prior knowledge on the largest eigenvalue of the inertia matrix. Attitude control with globally stable closed-loop dynamics employing the Euler parameters was also discussed in Ref. 8. The main result in that paper was derived from a feedback-linearization-like approach, wherein linear attitude error dynamics was prescribed and the control law was derived to enforce the linear error dynamics. The feedback control law assumed the knowledge of the attitude quaternion and the angular velocity information for implementation.

The fundamental contribution of this Note is the derivation of a new class of PI controllers for the attitude stabilization problem without using explicit velocity feedback. This work combines aspects of the passivity filter formulations together with the choice of a Lyapunov function containing cross/mixed terms involving the various states. The requirement for such cross terms is dictated by the presence of integral feedback terms within the control law that cannot otherwise be accommodated within the conventional passivity filters. As will be illustrated through the subsequent discussions, the important consequence is that we are able to guarantee global asymptotic stability for the nonlinear closed-loop dynamics in the ideal case (zero external disturbances) without requiring any sort of prior information on the body inertia matrix. Further, through numerical simulations we will evaluate potential advantages derived through the inclusion of the integral feedback term within the control law by computing the attitude error convergence in the presence of unknown (constant) disturbance torques. Local disturbance rejection is illustrated through a linearization about the nominal equilibrium point.

Presented as Paper 2004-202 at the AAS/AIAA 14th Space Flight Mechanics Meeting, Maui, Hawaii, 8–12 February 2004; received 17 February 2004; revision received 1 June 2004; accepted for publication 1 June 2004.

*Assistant Professor, Department of Mechanical Aerospace Engineering; subbarao@mae.uta.edu. Member AIAA.
†Assistant Professor, Department of Aerospace Engineering and Engineering Mechanics; makella@mail.utexas.edu. Member AIAA.